A SET OF FOUR INDEPENDENT POSTULATES FOR BOOLEAN ALGEBRAS*

BY

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In these Transactions, October, 1913, Sheffer† presented the following set of independent postulates for Boole's logic:

I. There are at least two distinct K-elements.

II. Whenever a and b are K-elements, $a|b$ is a K-element.

Definition. $a' = a|a$.

III. Whenever a and the indicated combinations of a are K-elements, $(a')' = a$.

IV. Whenever a, b, and the indicated combinations of a and b are K-elements, $a|(b|b') = a'$.

V. Whenever a, b, c, and the indicated combinations of a, b, and c are K-elements, $[a|(b|c)]' = (b'|a)|(c'|a)$.

This is the most economical set of postulates that has so far been proposed for Boolean algebras. Not only is the number of primitive propositions considerably smaller than that of the smallest set‡ of previous date—five instead of nine—but also the special elements "zero," the "whole," and the "negative" are all defined and their properties deduced. This economy Dr. Sheffer effected by basing the algebra on the powerful operation of "rejection," $|$.§ Choosing the primitive ideas that Dr. Sheffer chose, I give below a set of four independent postulates from which his five Postulates I–V are easily deduced.

New set of four postulates for Boolean logic

If we take *class* K and *operation* $|$ as primitive ideas,|| Boolean logic can be deduced from the following postulates:

* Presented to the Society, August 3, 1915.

† H. M. Sheffer: *A set of five independent postulates for Boolean algebras, with application to logical constants*, these Transactions, vol. 14 (1913), pp. 481–488.

‡ Professor E. V. Huntington's. See his *Sets of independent postulates for the algebra of logic*, these Transactions, vol. 5 (1904), pp. 288–309. See also the writer's *A complete set of postulates for the logic of classes, etc.*, University of California Publications in Mathematics, vol. 1, no. 4, pp. 87–96.

§ Interpreted concretely, the "reject" $a|b$ may be read: "that which is neither a nor b."

|| The symbol "$=$" is not taken as a primitive idea. By $a = b$, we agree to mean that a and b can be interchanged.

P_1. *K contains at least two distinct elements.*

P_2. *If a, b are elements of K, $a|b$ is an element of K.*

DEFINITION 1. $a' = a|a$.

P_3. *If a, b, and the combinations indicated are elements of K,*

$$(b|a)|(b'|a) = a.$$

P_4. *If a, b, c, and the combinations indicated are elements of K,*

$$a'|(b'|c) = [(b|a')|(c'|a')]'.$$

The following system satisfies all the Postulates P_1–P_4, and thus proves these postulates to be consistent:

K = a class of two elements e_1, e_2; $e_i | e_j$ = the element given by the following "rejection" table,

$\vert$	e_1	e_2
e_1	e_2	e_1
e_2	e_1	e_1

THEOREM 1. $(a')' = a$.

For, in P_4, let $b = c = a$. Then, by Definition 1 and P_3,* the left member becomes

$$a'|(a'|a) = (a|a)|(a'|a) = a;$$

and, by P_3, the right member becomes

$$[(a|a')|(a'|a')]' = (a')'.$$

DEFINITION 2. $a'' = (a')'$.

THEOREM 2. $a|(b|b') = a'$.

For, by 1, P_4, 1, and P_3,†

$$a|(b|b') = a''|(b''|b') = [(b'|a'')|(b''|a'')]' = [(b'|a)|(b''|a)]' = a'.$$

THEOREM 3. $[a|(b|c)]' = (b'|a)|(c'|a)$.

For, by 1, P_4, 1, 1,†

$$[a|(b|c)]' = [a''|(b''|c)]' = [(b'|a'')|(c'|a'')]''$$
$$= [(b'|a)|(c'|a)]'' = (b'|a)|(c'|a).$$

Since Sheffer's Postulates I, II are identical with Postulates P_1, P_2, and since Postulates III, IV, V correspond respectively to Theorems 1, 2, 3,—all of Sheffer's postulates, and hence all of Boolean logic, can be deduced from Postulates P_1–P_4. Indeed, the two sets of postulates are equivalent; for propositions P_1–P_4 can also be derived from I–V, as may be easily verified.

* The use of P_1 and P_2 will not be indicated.

† The use of Definition 2 is not indicated.

Independence of the four postulates

The following systems $\bar{P}_1$–$\bar{P}_4$ are such that $\bar{P}_i$ contradicts none of the Postulates P_1–P_4 except P_i, thus showing the independence of P_i from the rest of the postulates.

$\bar{P}_1$. K = a class of one element e; $e|e = e$.

$\bar{P}_2$. K = a class of two elements e_1, e_2; $e_i|e_j$ is given by the following table:

$\vert$	e_1	e_2
e_1	e_2	x ,
e_2	x	e_1

where x is not in K. P_2 is false for $a \neq b$.

$\bar{P}_3$. K = a class of two elements e_1, e_2; $e_i|e_j$ is given by the following table:

$\vert$	e_1	e_2
e_1	e_1	e_1 .
e_2	e_2	e_1

Here P_3 is false for $a = e_1$, $b = e_2$.

$\bar{P}_4$. K = a class of two elements e_1, e_2; $e_i|e_j$ is given by the following table:

$\vert$	e_1	e_2
e_1	e_1	e_2 .
e_2	e_1	e_2

Here P_4 is false for $a = e_1$, $c = e_2$.

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